Fracture Mechanics You Can Understand.

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This document is intended to present much of the theory of linear elastic fracture mechanics (LEFM) in a way that is readily understandable to a second year engineering student. For a given crack geometry in a given structure, material type and boundary conditions, LEFM provides the formula below for the normal stress $\sigma_{yy}(x,0)$ on the crack plane with the crack. See Fig. 1.

$$\sigma_{yy}(x,0) = \frac{K_I}{\sqrt{2\pi r}} f(\theta) \tag{1}$$

where K_I is the amplitude of the function $f(\theta)/(\sqrt{2\pi r})$. The function $f(\theta)/(\sqrt{2\pi r})$ is only a function of polar coordinates (r,θ). For a centre crack in an infinite plate with uniaxial tensile stress σ^*_{yy} before (without) the crack and crack length a, the value of the coefficient K_I, which is called the stress intensity factor, is computed from the equation below.

$$K_I = \sigma^*_{yy}\sqrt{\pi a} \tag{2}$$

where σ^*_{yy} is is the traction normal to the crack plane evaluated in the UNCRACKED specimen or structure. Note that σ^*_{yy} and $\sigma_{yy}(x,0)$ are very different things.

LEFM succeeded because experiments showed that if for a given material, crack geometry and load case, K_I was less than a critical value called the fracture toughness that is denoted K_{IC}, the crack was unlikely to grow rapidly and the risk the structure would fail was low. However, if K_I was greater than the fracture toughness, the risk that the crack would grow rapidly and the structure would fail was high. It is important to note that K_{IC} is not a material property but it is a function of the material, structure geometry, crack geometry and load case. In other words, it is a function of the design.

The following sections will show several examples that demonstrate that the associated stress distributions in the crack plane that are computed from LEFM theory and computed directly with FEM are in some sense equivalent.

1 A Center-Cracked Plate

In this example of a center-cracked plate of finite width b and σ^*_{yy} is the tensile stress on the crack plane without the crack, the value of the stress intensity factor K_I for plane strain deformation is given by the equation below.

$$K_I = \sigma^*_{yy}\sqrt{\pi a\left(\frac{2b}{\pi a}tan\frac{\pi a}{2b}\right)} \tag{3}$$

Now the constant K_I is not a fundamental constant like the speed of light. It is only a constant for this particular crack geometry, material and this particular stress state in this particular structure or global body. Thousands of papers have been written to try to provide an equation to evaluate K_I for every possible crack geometry and loading situation. A typical textbook on fracture mechanics will have many such equations, each for a different crack geometry, structure geometry and load case. Thousands of theses and tens of thousands of papers have been devoted to LEFM. Just as one needs a contact information for each individual person, one needs a different equation for the stress intensity factor for each variation of a crack geometry, crack location, structure design and material.

For a crack plane normal to the y-axis that is loaded with a uniaxial yy-stress, the yy-stress component near the crack tip as a function of polar coordinates, i.e., as a function of distance r from the crack tip and angle θ from the crack plane, is denoted $\sigma_{yy}(r, \theta)$. Its value is given by the following equation.

$$\sigma_{yy}(r, \theta) = \frac{K_I}{\sqrt{2\pi r}}\, cos\frac{\theta}{2}\left[1 - sin\frac{\theta}{2}sin\frac{3\theta}{2}\right] \tag{4}$$

This equation is a useful approximation in a region near the crack tip. It is like Goldilock's porridge, i.e., not too close and not too far. As the distances to the crack tip approaches zero, the stress in the equation goes to infinity. This is clearly not possible in real materials that cannot have an infinite yield stress. Advocates of LEFM rationalize this by arguing that for sufficiently brittle materials, the region under going plastic strain is small and the error caused by ignoring the error in this plastic region is acceptable. Therefore this equation could only be realistic in the part of the curve for $r > \epsilon$ some distance from the crack tip where the value of the stress declines with distance from the crack tip at the rate predicted by the equations in LEFM theory. In real problems at stresses above the yield stress, the material will yield plastically. In no real case will the stress go to infinity. In LEFM theory, the stress does go to infinity at the crack tip. At distances far from the crack tip, the stress in this equation goes to zero. This is also not realistic. The stress near a small crack should decay to the stress σ^*_{yy} in the uncracked structure for distances sufficiently far from the crack, i.e., as $r \mapsto \infty$. Thus the stress computed by LEFM is only a useful approximation in the region not too close to the crack tip and not too far from the crack tip.

What is the basis of this theory? Who developed it and why? The answer is that some of world's brightest applied mathematicians solved some very difficult mathematical problems to compute the eigenvalues and eigenfunctions associated with the singularity in the stress state near the crack tip. The stress intensity factor K_I is the real part of the complex eigenvalue with the lowest non-zero real part in this particular problem. It is a bit like the first term in a Taylor series or the critical stress for Euler's first buckling mode for a slender cylinder loaded in compression. However, to my knowledge, engineers who use LEFM never compute any higher order eigen values or eigenfunctions.

So what is the justification for using a theory that is not realistic? Well all theories are based on a set of assumptions. In science and engineering there are no theories of everything. It is the user's responsibility to decide if the assumptions in the theory are sufficiently realistic, i.e., what to leave in and what to leave out. To be more precise it is the users responsibility to decide if the theory or mathematical model when used appropriatedly provides predictions and information that would be useful in making a decision. Clearly, there is no question that LEFM has been very useful.

Why did people accept the unrealistic assumptions? The only reason that we can image is that people did not want or were not capable of analyzing the stress in a complex body that had a crack. In the period from 1950 to 1970 in which LEFM was developed and first used in engineering, LEFM was their best option. This is no longer true. Using modern FEM packages would be much cheaper, faster, more accurate, more realistic, more general and easier to understand and would require far less training. Fracture mechanics would be viewed as a standard mechanics problems requiring only a knowledge of standard fundamental principles of mechanics and a knowledge of the required properties of the materials. Instead of computing only the first eigen pair as in LEFM, FEM would compute all of functions that project the exact solution of the infinite dimensional mathematical problem onto the function space defined by the FEM elements in the finite element mesh near the crack tip. To get a higher order approximation just refine the FEM mesh or replace lower order elements with higher order elements.

Although LEFM theory will become irrelevant in modern engineering, just as the LATIN language and drafting are, the full mathematical theory that solves for complex eigen functions and eigen values could be a powerful tool for people who have the mathematical expertise to understand and exploit it for applications that would benefit from this theory. Others are advised not to go near the full mathematical theory until they have sufficient understanding of the mathematics.

2 Replacing Linear Elastic Fracture Mechanics

The only reason for using LEFM that the authors can imagine is to avoid a global stress analysis of a large and possibly complex structure that has the crack. The global stress analysis of the large and possibly complex structure without the crack had to be done to get the stress state in the neighbourhood of the crack before the crack formed. Decades ago, before FEM codes were developed and reached their current level of maturity, it was very expensive to compute even a crude estimate of the stress in the un-cracked complex structure such as a ship. I cannot imagine how one could compute the stress in a complex structure such as ship with even one crack using the classical theory of mechanics. I suspect that this is exactly why LEFM was developed. It provided a tractable method of estimating the stress state near a crack in a complex structure. The theory was complicated and there were serious limitations to the theory but at the time it clearly was the best option available.

Since that time FEM codes have been developed and reached a high level of maturity. Since domain decomposition algorithms are now available that enable a stress analysis to be done quickly, cheaply and almost automatically using FEM in a subdomain that is only large enough to contain the region in which the stress changes introduced by the crack are significant. The authors argue that it is easier, simpler and at least as accurate to compute the stress distribution near the crack tip using FEM on as fine a mesh as one deems useful. This would avoid a great deal of mathematical gymnastics, the highly restrictive assumptions of doubtful validity that are assumed in LEFM and would be much easier for design engineers to understand and to use in optimizing real designs. Since very few design engineers actually understand the theory or the assumptions that underly LEFM, this seems highly desirable. Moreover, the FEM analysis can use nonlinear models of the material and geometry that are more realistic.

What must be retained from the past 70 or years of research on fracture mechanics is the experimental data. The weaknesses and limitations of LEFM do not in any way diminish the value of the data acquired in the vast number of experiments for fracture mechanics. This experimental data completely retains its value and remains essential to the fracture mechanics analysis that will be done using FEM analysis. Not only could the FEM computer models could be correlated more strongly with the experimental data, but they could be used to design optimal experiments.

3 Measuring K_{IC} - Fracture Mechanics Tests

Designers would like to have the opportunity to conduct experiments quickly and at low cost. Ideally the experiment would use the real structure or a prototype. In the early stages design, this is not possible because nothing exists but concepts. Then test specimens must be designed.

This usually creates a need to design optimal experiments. Often one seeks to optimize the specimen size. Smaller experiments could have a lower cost until the size reaches a point at which the cost begins to climb. For example a standard Charpy Impact test specimen is $1 \times 1 \times 10$ cm which is usually considered the optimum size for a toughness test of a material such as steel. There are a few wide plate tensile test machines in the world that test very large specimens such 1 m diameter pipes width 25 mm thick walls of high strength steel but the machines and the specimens are expensive. However, when one does not have material in sizes large enough to make the desired test specimen, one could be pushed to devise tests with specimens of smaller size. This has been done in the nuclear industry where only very small specimens are sometimes available.

The risk of fracture is higher in larger structures for several reasons. The probability of more and larger defects is larger. The stress state is more highly constrained which favours lower toughness fracture modes. The quality of the microstructure is usually lower in larger structures because of difficulties and limitations in the manufacturing process. Devising fracture toughness tests for very large structures such as nuclear reactor pressure vessels that can weigh hundreds of tonnes is a challenge. One of the challenges is that the fracture mode can be sensitive to whether the stress state is plane strain or plane stress. Plane strain stress state has a lower fracture toughness. This is the reason that it is often difficult to relate data from Charpy tests to the toughness of a large structure made with essentially the same material. This has led to the design of special tests to measure fracture toughness. One of the most popular is the compact tension test (CT). CT tests are often made in a size with thickness close to the thickness of the structure being designed. Designing smaller test specimens is an active research area.

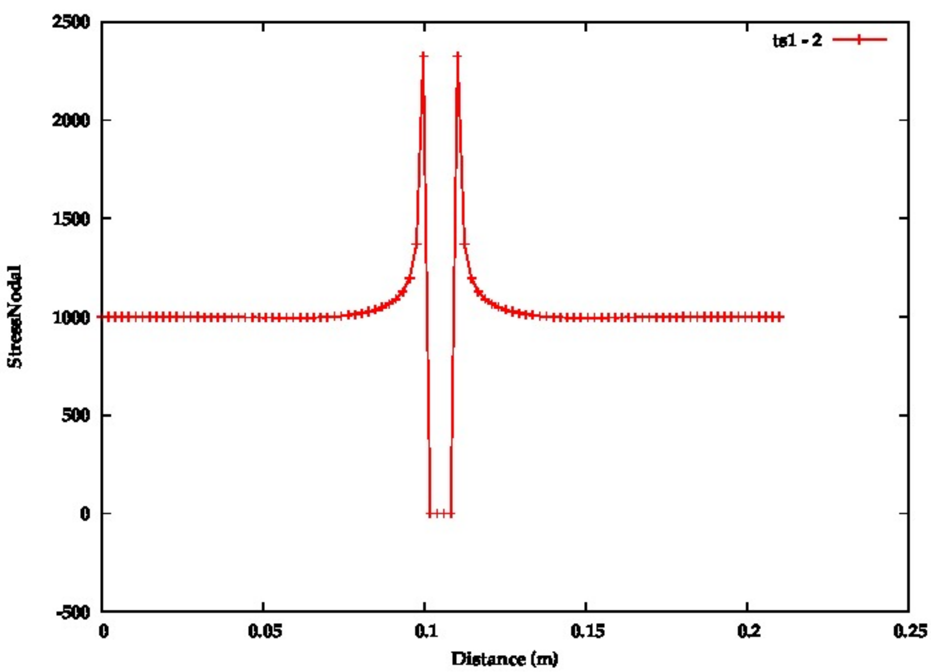

Figure 1: An FEM solution of the yy-stress component in a center cracked plate with a uniaxial plane strain tensile loading without the crack. Without the crack, the uniaxial tensile stress would be 1000 Pa

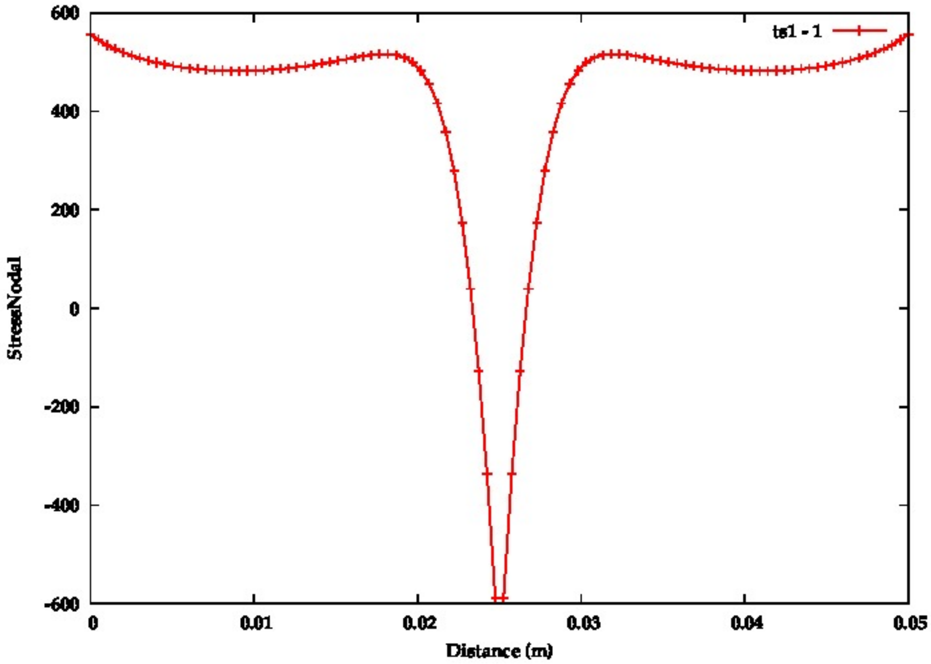

Figure 2: An FEM solution of the xx-stress component in a center cracked plate with a uniaxial plane strain tensile loading without the crack.

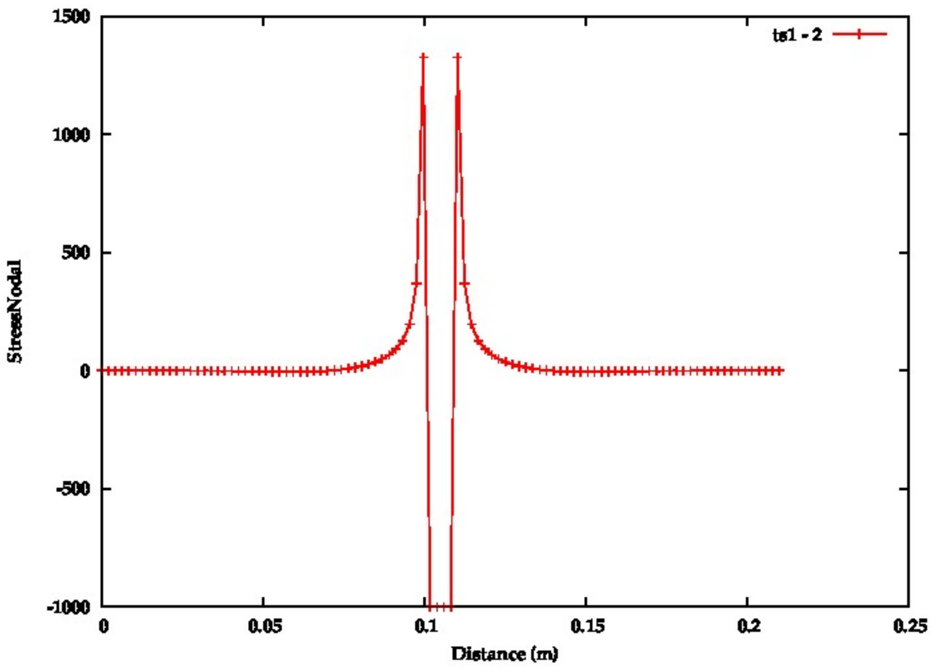

Figure 3: An FEM solution of the yy-stress component in a center cracked plate with zero uniaxial plane strain tensile loading with traction from the crack plane in the uncracked plate applied to the crack. Compare to Fig. 1.

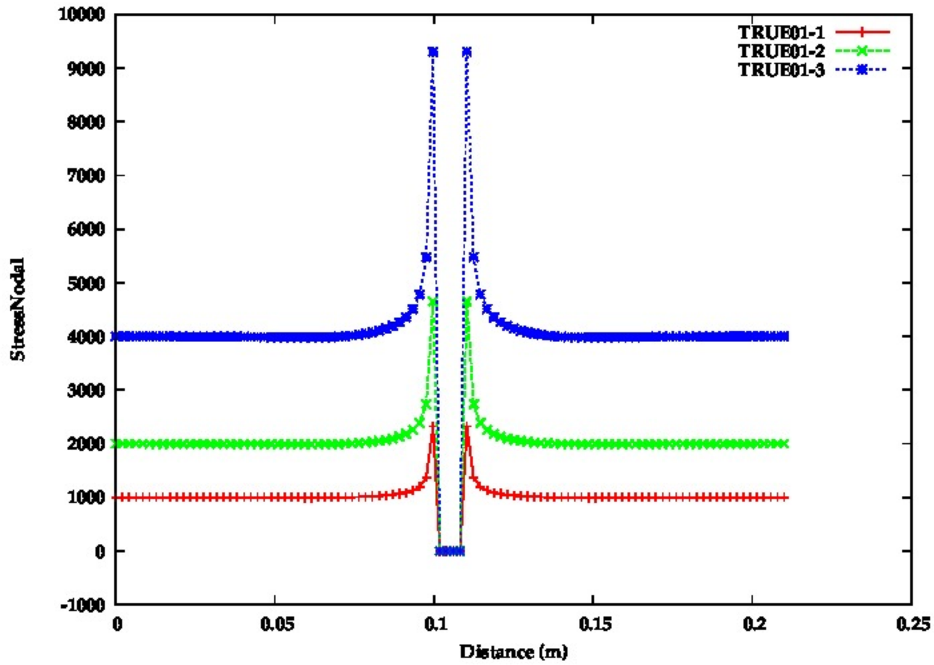

Figure 4: An FEM solution of the yy-stress component in a center cracked plate. Without the crack it would be uniaxial plane strain tensile loading of 1000, 2000 and 4000 MPa. Compare to Fig. 1.

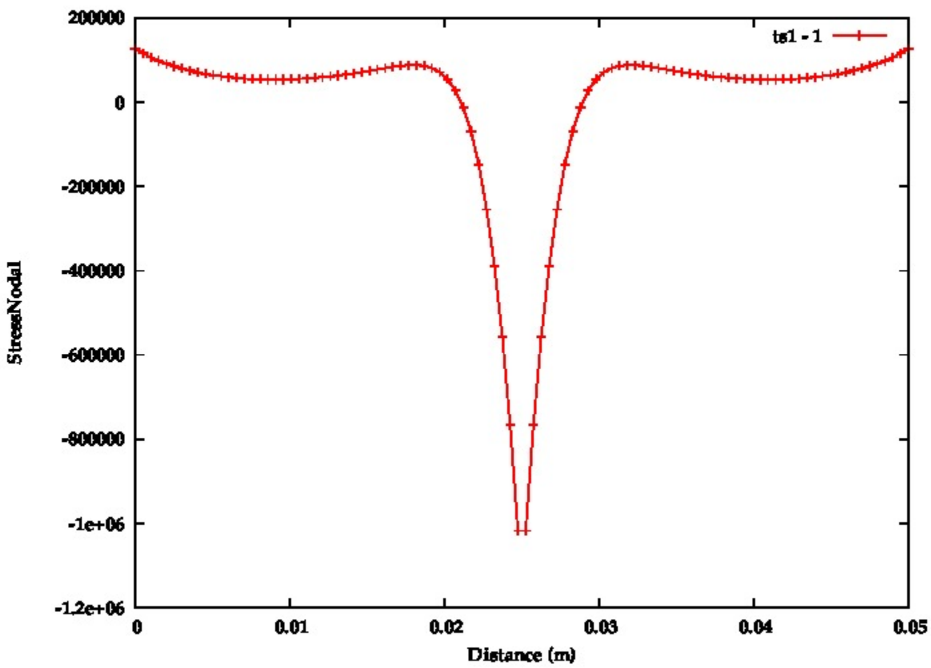

Figure 5: An FEM solution of the xx-stress component in a center cracked plate. Without the crack it would be uniaxial plane strain tensile loading.

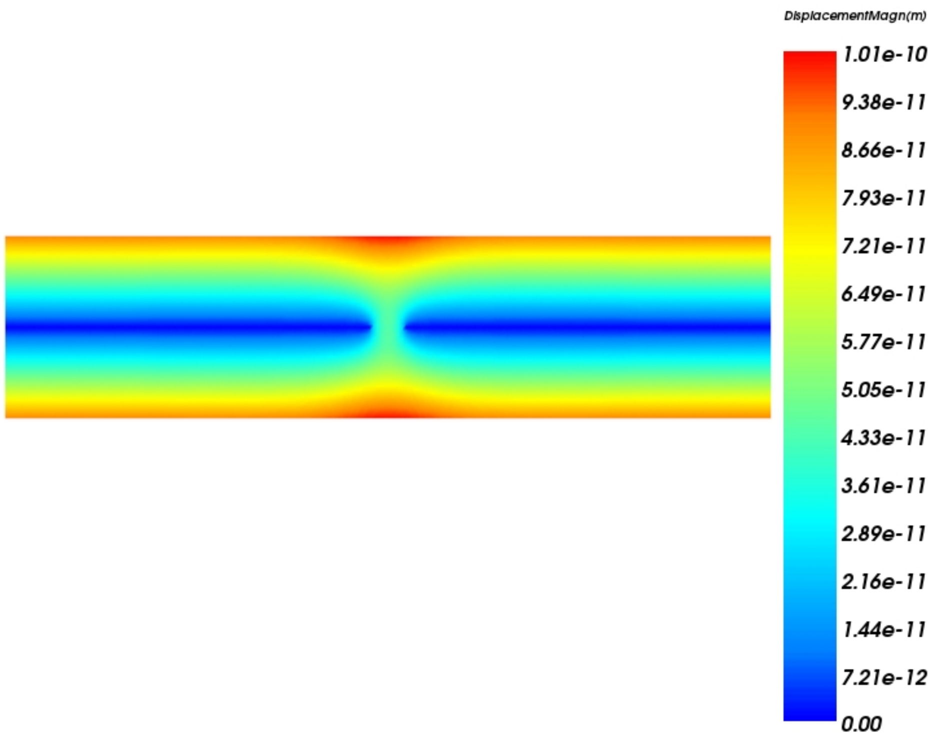

Figure 6: An FEM solution of the y displacement component in a center cracked plate with zero uniaxial plane strain tensile loading with traction from the crack plane in the uncracked plate applied to the crack.

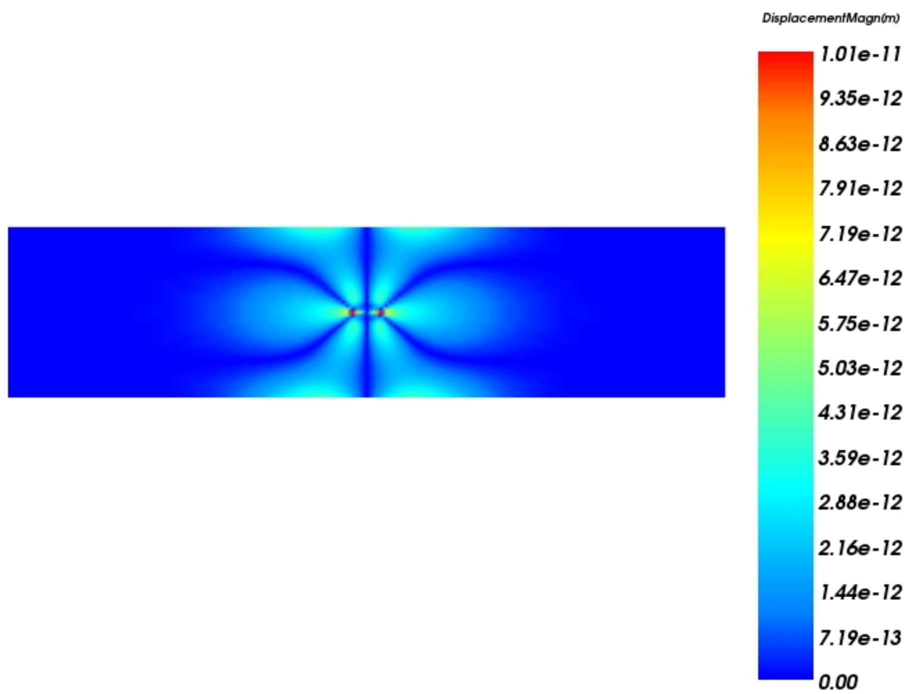

Figure 7: An FEM solution of the x displacement component in a center cracked plate with zero uniaxial plane strain tensile loading with traction from the crack plane in the uncracked plate applied to the crack.

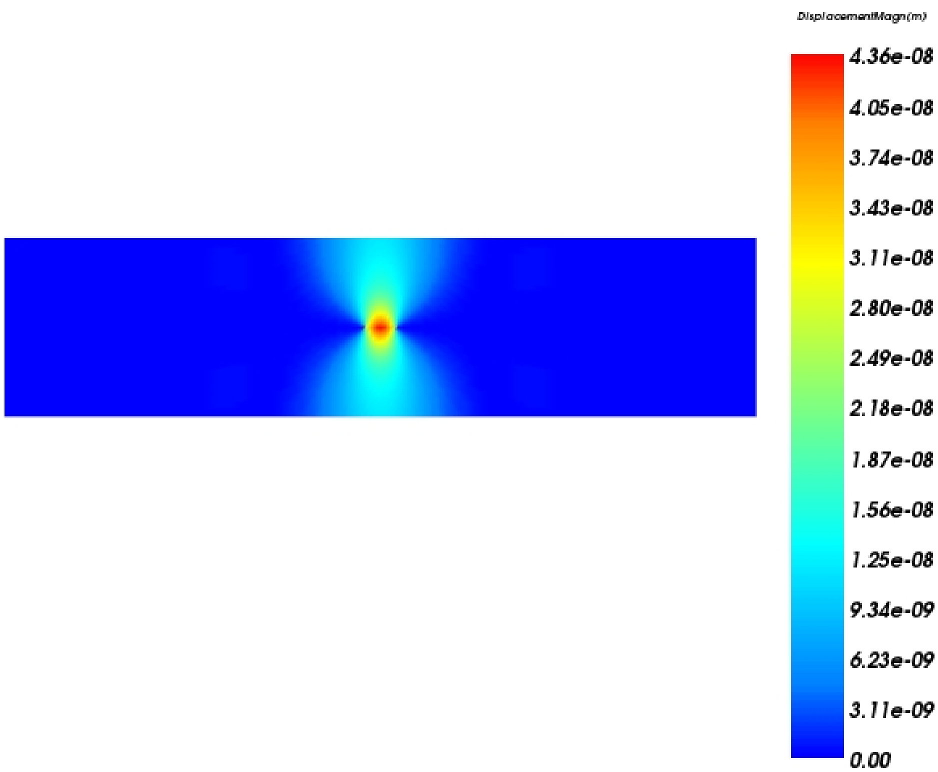

Figure 8: An FEM solution of the y displacement component in a center cracked plate with zero uniaxial plane strain tensile loading with traction from the crack plane in the uncracked plate applied to the crack.

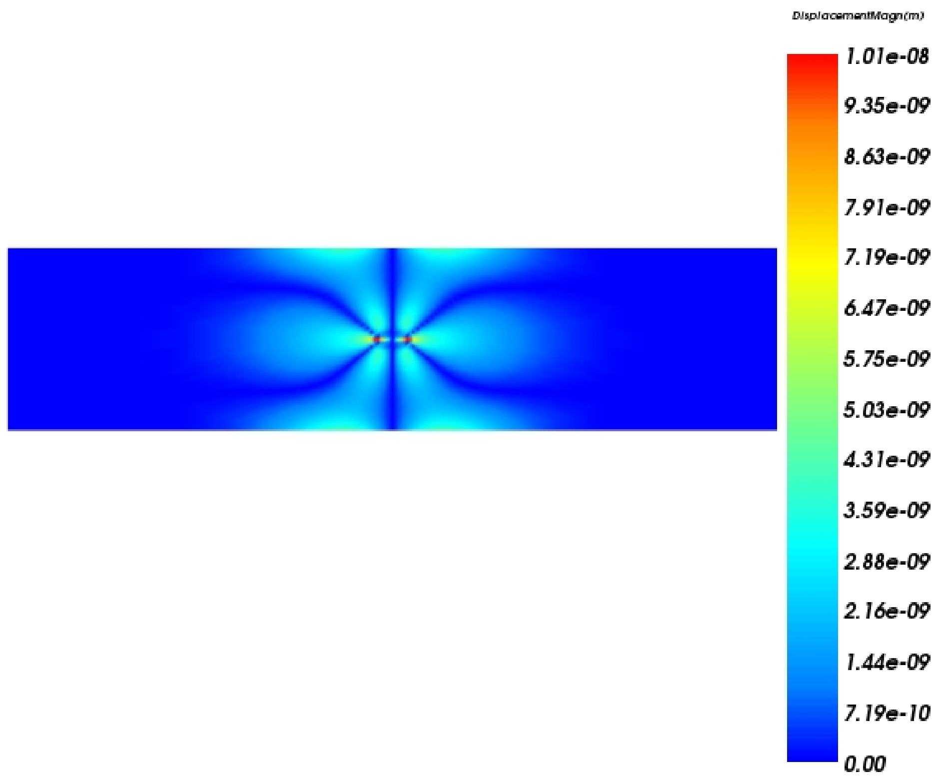

Figure 9: An FEM solution of the x displacement component in a center cracked plate with zero uniaxial plane strain tensile loading with traction (0, 1000, 0) the crack.

Figure 10: A plot of the yy-stress component in a center cracked plate with a uniaxial plane strain tensile in the plane of the crack. if the part of the plot to the right of the peak is plotted as a function of $1/\sqrt{r}$, a part would be the singular function that is approximated in linear elastic fracture mechanics. This FEM analysis is closer to reality.

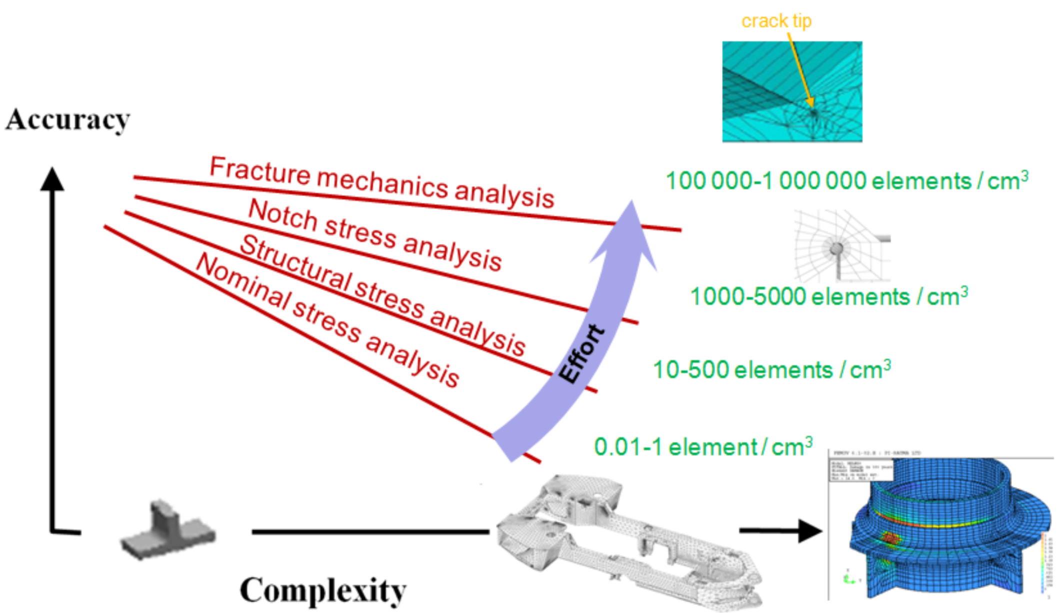

Figure 11: Illustration of different techniques that can be used to solve fatigue related issues. Marquis (2009)

References

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[3] Felipe P. Calderon, Nobuo Sano, Yukio Matsushita, Diffusion of manganese and silicon in liquid iron over the whole range of composition, Metallurgical and Materials Transactions B, December 1971, Volume 2, Issue 12, pp 3325-3332